Mid-Term Exam
STAT 501 (Term 231)

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Exercise 1. Let X be a random variable with the following c.d.f

$$F(x) = \tan(x)\mathbf{1}_{(0,\frac{\pi}{8})} + ((x-\tfrac{\pi}{8})^2 + \tfrac{1}{2})\mathbf{1}_{[\frac{\pi}{8},\frac{\pi}{4})} + \mathbf{1}_{[\frac{\pi}{4},+\infty)}$$

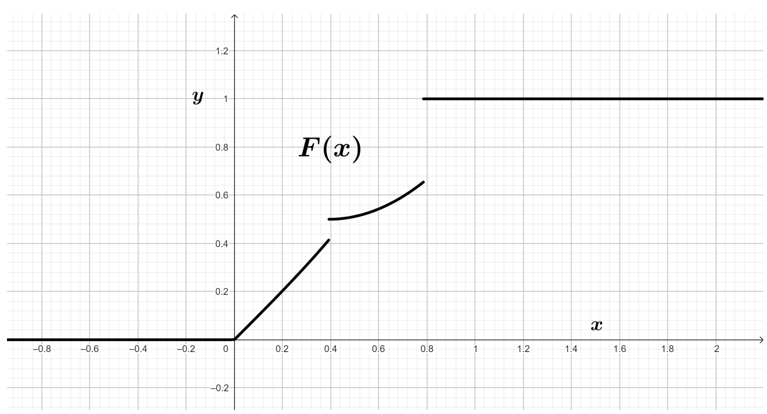

Figure 0.1: The plot of F

You can use without proof $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$.

1. What is the type of X? X is mixed (continuous and discrete).
2. What is the support of X? $(0, \frac{\pi}{4}]$ or $[0, \frac{\pi}{4}]$
3. Write down the point masses of X as well as their probabilities (in case of existence). Point masses correspond to the jumps in the c.d.f. That is, they are $\frac{\pi}{8}, \frac{\pi}{4}$.
4. Find the law of X, $dF(x)$.

$$dF(x) = \sec^2(x)\mathbf{1}_{(0,\frac{\pi}{8})} + 2(x - \frac{\pi}{8})\mathbf{1}_{(\frac{\pi}{8},\frac{\pi}{4})} + \underbrace{(\frac{1}{2} - (\sqrt{2}-1))}_{\text{jump at } \frac{\pi}{8}}\delta_{\frac{\pi}{8}} + \underbrace{(1 - ((\frac{\pi}{4} - \tfrac{\pi}{8})^2 + \tfrac{1}{2}))}_{\text{jump at } \frac{\pi}{4}}\delta_{\frac{\pi}{4}}$$

5. Explain why the expectation of X, $\mathbf{E}(X)$ is well defined. It is because the continuous part of support is bounded and the discrete part is finite.

6. Find the value of $\mathbf{E}(X)$.

$$\mathbf{E}(X) = \int_0^{\frac{\pi}{4}} x dF(x) = \int_0^{\frac{\pi}{4}} x \sec^2(x)dx + \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} x \cdot 2(x - \frac{\pi}{8})dx + (\frac{1}{2} - (\sqrt{2}-1)) \cdot \frac{\pi}{8} + (1 - ((\frac{\pi}{4} - \frac{\pi}{8})^2 + \frac{1}{2})) \cdot \frac{\pi}{4}$$

7. Find the quantile function q of F.

$$q(u) = \begin{cases} \arctan(u) & 0 < u < \frac{1}{2} - (\sqrt{2}-1) \\ \frac{\pi}{8} & \frac{1}{2} - (\sqrt{2}-1) \leq u < \frac{1}{2} \\ \sqrt{u - \frac{1}{2}} + \frac{\pi}{8} & \frac{1}{2} \leq u < 1 - ((\frac{\pi}{4} - \frac{\pi}{8})^2 + \frac{1}{2}) \\ \frac{\pi}{4} & 1 - ((\frac{\pi}{4} - \frac{\pi}{8})^2 + \frac{1}{2}) \leq u < 1 \end{cases}$$

8. What is the distribution of $q(U)$ where U is uniformly distributed on $(0,1)$? The same distribution as X.

Exercise 2. Let X, Y be two independent random variables geometrically distributed according to some parameter $p \in (0,1)$. Let $Z = \min(X, Y)$.

1. What is $P(\{X = n\})$? By definition $P(\{X = n\}) = p(1-p)^{n-1}$.

2. Deduce $P(\{X \geq n\})$.
   Set $q = 1 - p$. Then

$$P(X \geq n) = \sum_{k \geq n} P(X = k) = \sum_{k \geq n} q^{k-1} p = \frac{q^{n-1} p}{1-q} = q^{n-1}.$$

3. Show that $P(\{Z \geq n\}) = q^{2n-2}$. By independence of X and Y we have $P(\{Z \geq n\}) = P(\{X \geq n\})P(\{Y \geq n\})$. Thus

$$P(Z \geq n) = P(X \geq n)P(Y \geq n) = q^{2n-2}.$$

4. Deduce the value of $P(\{Z = n\})$. It follows from $\{Z \geq n\} = \{Z = n\} \uplus \{Z \geq n+1\}$. Hence

$$P(Z = n) = P(Z \geq n) - P(Z \geq n+1) = q^{2n-2} - q^{2n} = q^{2n-2}(1 - q^2).$$

5. Identify the law of Z. From the expression of $P(\{Z = n\})$, Z is geometrically distributed with parameter $1 - q^2$.

Exercise 3. Let $V = (X, Y)$ be a random vector uniformly distributed on the Δ, the depicted planar triangle.

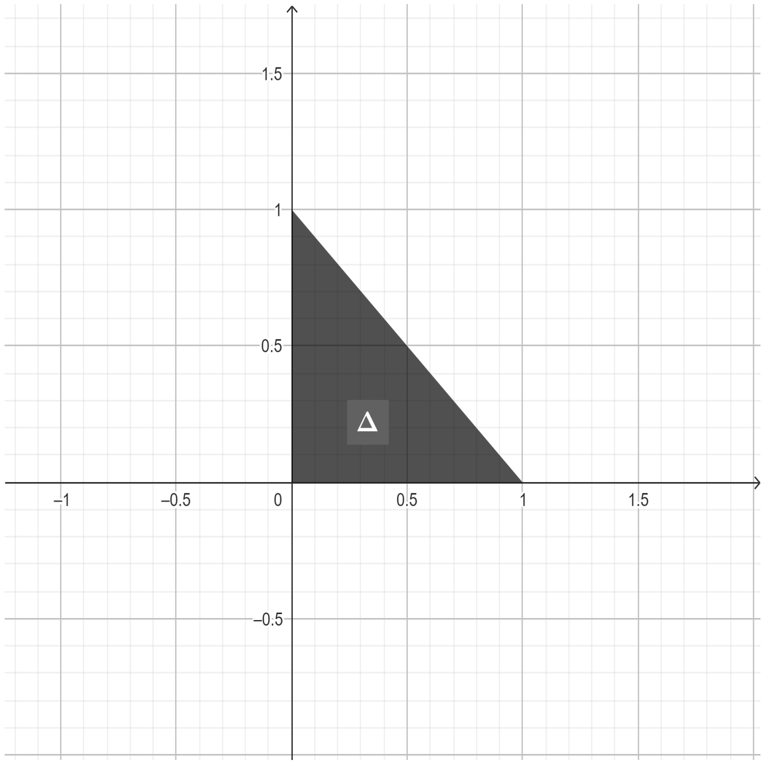

Figure 0.2: The vertices of the triangle Δ are $(0,0),(1,0),(0,1)$.

1. What is the joint p.d.f of V, $\rho(x,y)$? $\rho(x,y) = \frac{1}{\text{area}(\Delta)}\mathbf{1}_\Delta(x,y) = 2\mathbf{1}_\Delta(x,y)$.

2. Show that the marginal distribution of X is $\rho_X(x) = 2(1-x)\mathbf{1}_{(0,1)}$. For $x \in (0,1)$ we have

$$\rho_X(x) = \int \rho(x,y)dy = \int_0^{1-x} 2dy = 2(1-x).$$

3. Find the expectation of X.

$$\mathbf{E}(X) = \int_0^1 x\rho_X(x)dx = \int_0^1 x \cdot 2(1-x)dx = \frac{1}{3}.$$

4. Find the c.d.f of X.

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x \rho_X(t)dt = 2x - x^2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

5. On the given figure, shadow the region $R := [\frac{1}{2}, +\infty) \times [\frac{1}{2}, +\infty)$.

6. Deduce that $P(\{V \in R\}) = 0$. This is because R is outside the support of V, Δ.

7. Are the random components of V independent?
First method : If they are independent then

$$P(\{V \in R\}) = 0 = P(\{X \in [\frac{1}{2}, +\infty)\})P(\{Y \in [\frac{1}{2}, +\infty)\})$$

but none of $P(\{X \in [\frac{1}{2}, +\infty)\})$ or $P(\{Y \in [\frac{1}{2}, +\infty)\})$ is zero.
Second method : X and Y have the same distribution (by the symmetry of the triangle). So if they are independent we must have

$$\rho(x, y) = 2\mathbf{1}_{\Delta}(x, y) = 2(1 - x)\mathbf{1}_{\{x \in (0,1)\}} 2(1 - y)\mathbf{1}_{\{y \in (0,1)\}}$$

which is not correct.