KING FAHD UNIVERSITY OF PETROLEUM & MINERALS, DHAHRAN, SAUDI ARABIA
DEPARTMENT OF MATHEMATICS

STAT 502: Statistical Inference

Term 211, First Major Exam, Saturday October 16, 2021, 09:00 AM

Name: Solution ID #: ____________________

Q1: For a random sample of size n drawn from the probability density function:

$$f(x) = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}};\ 0 < x < \beta.$$

Assuming β to be a known constant, derive the Maximum Liklihood Estimator (MLE) for the unknown parameter α. Also verify whether the likelihood function has been maximized at the value of $\hat{\alpha}$ that you chose.

$$f(x) = \alpha \beta^{-\alpha} x^{\alpha-1} \quad ; \quad 0 < x < \beta$$

$$L(x;\alpha) = \alpha^{n} \beta^{-n\alpha} \prod_{i=1}^{n} x_i^{\alpha-1}$$

$$l = \ln L(x;\alpha) = n \ln \alpha - n\alpha \ln\beta + (\alpha-1)\sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial}{\partial \alpha} l = \frac{n}{\alpha} - n\ln\beta + \sum_{i=1}^{n} \ln x_i$$

Set it equal to zero

$$\frac{n}{\hat{\alpha}} = n\ln\beta - \sum_{i=1}^{n} \ln x_i$$

$$\hat{\alpha} = \frac{n}{n\ln\beta - \sum_{i=1}^{n} \ln x_i}$$ which is the MLE of α.

To verify that it maximizes the log liklihood

$$\frac{\partial^2}{\partial \alpha^2} l = -\frac{n}{\alpha^2}$$

Substitute $\hat{\alpha}$ in second derivative we get

$$\left.\frac{\partial^2}{\partial \alpha^2} l \right|_{\hat{\alpha}} = \frac{-n}{\left[n / \left(n \ln\beta - \sum_{i=1}^{n} \ln x_i \right) \right]^2}$$

$$= -\frac{\left(n \ln\beta - \sum \ln x_i \right)^2}{n}$$

$$= -\frac{\left[\sum_{i=1}^{n} \left(\ln\beta - \ln x_i \right) \right]^2}{n}$$

which is always negative so the log liklihood is maximized at $\hat{\alpha}$.

Q2: Age of the car is an important factor for the car buyers. Suppose that the age of car is uniformly distributed over the interval $(\theta - 1, \theta + 2)$, where θ is the unknown parameter. Suppose that $Y_1, Y_2, \ldots \ldots, Y_n$ denote a random sample of car ages.

a) Show that $\bar{Y}$ is a biased estimator of θ. Also compute the Bias of $\bar{Y}$.
b) Using mean square error, show that $\bar{Y}$ is inconsistent with θ.
c) Find a function of $\bar{Y}$ that is an unbiased estimator of θ. Is your new estimator consistent with θ? For checking consistency, use the same method as in part (b).

Note: For a Uniform random variable X, $f(x) = \frac{1}{b-a}$; $a < x < b$. $E(X) = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

$$f(y_i) = \frac{1}{\theta+2-\theta+1} = \frac{1}{3}; \quad \theta-1 < y < \theta+2$$

$$E(y_i) = \frac{\theta+2+\theta-1}{2} = \theta + 0.5$$

$$E(\bar{Y}) = E\left(\frac{Y_1 + Y_2 + \cdots + Y_n}{n}\right)$$

$$= \frac{E(Y_1) + E(Y_2) + \cdots + E(Y_n)}{n}$$

$$= \frac{(\theta+0.5)+(\theta+0.5)+\cdots+(\theta+0.5)}{n} = \frac{\cancel{n}(\theta+0.5)}{\cancel{n}}$$

$$E(\bar{Y}) = \theta + 0.5 \neq \theta$$

which proves that $\bar{Y}$ is biased for θ.

$$E(\bar{Y}) - \theta = Bias(\bar{Y}) = 0.5$$

Now $$Var(\bar{Y}) = \frac{Var(Y)}{n} = \frac{(b-a)^2}{12n} = \frac{(\cancel{\theta}+2-\cancel{\theta}+1)^2}{12(n)}$$

$$Var(\bar{y}) = \frac{9}{12n}$$

$$MSE(\bar{y}) = Var(\bar{y}) + [Bias(\bar{y})]^2$$

$$MSE(\bar{y}) = \frac{9}{12n} + \left(\frac{1}{2}\right)^2 = \frac{9}{12n} + \frac{1}{4}$$

$$\lim_{n \to \infty} MSE(\bar{y}) = \lim_{n \to \infty} \left(\frac{9}{12n} + \frac{1}{4}\right) = \frac{1}{4} \neq 0$$

$\Rightarrow \bar{y}$ is inconsistent.

Define $\hat{\theta} = \bar{y} - 0.5$

$$E(\hat{\theta}) = E(\bar{y}) - 0.5 = (\theta + 0.5) - 0.5$$

$$E(\hat{\theta}) = \theta$$

$$\Rightarrow Bias(\hat{\theta}) = 0$$

$$Var(\hat{\theta}) = Var(\bar{y} - 0.5) = Var(\bar{y}) = \frac{9}{12n}$$

$$MSE(\hat{\theta}) = \frac{9}{12n} + 0^2$$

$$\lim_{n \to \infty} MSE(\hat{\theta}) = \lim_{n \to \infty} \frac{9}{12n} = 0$$

$\Rightarrow \hat{\theta}$ is consistent with θ.

Q3: Suppose $X_1, X_2, \dots, X_n$ denote a random sample from Geometric distribution with density function $f(x_i) = \theta(1-\theta)^{x-1}; x = 1,2, \dots$. Derive a sufficient estimator of $\cancel{1}/\theta$. Is your estimator unbiased? If not, suggest a linear function of sufficient estimator that is also unbiased.

$$L(x;\theta) = \theta^n (1-\theta)^{\Sigma x_i - n}$$

$$= g(\hat{\theta}, \tfrac{1}{\theta}) \cdot h(x_i)$$

where $g(\Sigma x_i, \frac{1}{\theta}) = \theta^n (1-\theta)^{\Sigma x_i - n}$

and $h(x_i) = 1$

$\Rightarrow \Sigma x_i$ is sufficient for $1/\theta$.

Define $\hat{\theta} = \dfrac{\Sigma x_i}{n}$

$$E(\hat{\theta}) = \frac{1}{n} \Sigma E(x) = \frac{1}{n} \Sigma \left(\frac{1}{\theta}\right)$$

$$E(\hat{\theta}) = \frac{\cancel{n}}{\cancel{n}} \left(\frac{1}{\theta}\right) = \frac{1}{\theta}$$

$\Rightarrow \hat{\theta}$ is unbiased for $\frac{1}{\theta}$.

Q4: Suppose $X_1, X_2, \ldots, X_n$ denote a random sample from the probability density function:

$f(x) = \frac{2x}{\alpha} e^{-\frac{x^2}{\alpha}}; x > 0.$

Find minimum variance unbiased estimator of α. Also find the variance of your estimator.

$$L(x;\alpha) = 2^n \alpha^{-n} \prod x_i \; e^{-\frac{1}{\alpha}\sum x_i^2}$$

$$l = \ln L(x;\alpha) = n\ln 2 - n\ln\alpha + \sum \ln x_i - \frac{1}{\alpha}\sum x_i^2$$

$$\frac{\partial}{\partial\alpha} l = 0 - \frac{n}{\alpha} + \frac{1}{\alpha^2}\sum x_i^2$$

$$= \frac{n}{\alpha^2}\left(-\alpha + \frac{\sum x_i^2}{n}\right)$$

$$\frac{\partial}{\partial\alpha} l = \frac{n}{\alpha^2}\left(\frac{\sum x_i^2}{n} - \alpha\right)$$

$$= A(\hat{\alpha} - \alpha)$$

where $\hat{\alpha} = \frac{\sum x_i^2}{n}$

and $Var(\hat{\alpha}) = \frac{1}{A} = \frac{\alpha^2}{n}$.

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Good Luck