KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DEPARTMENT OF MATHEMATICS

STAT 502: Statistical Inference

Term 222, Midterm Exam
Saturday March 18, 2023, 06:00 PM

Name: ________________________________ ID #: ____________________

Question No	Full Marks	Marks Obtained
1	**30**	
2	**15**	
3	**09**	
4	**10**	
5	**06**	
Total	**70**	

Instructions:

1. Mobiles are not allowed in exam. If you have your **mobile** with you, **turn it off** and put it **on the table** so that it is visible to proctor.

2. Show all the derivation steps. There are points for the steps so if your miss them, you lose points. For multiple choice type questions, showing calculation steps is not required.

3. Derive every result that you use in your solution, unless mentioned otherwise.

Code 1

Q1: (3x10 = 30 pts.) Multiple choice questions.

(001) Let $X_1, X_2, \dots, X_n$ be a random sample from Normal distribution with mean μ and variance σ^2. Which one of the following is correct expression for the mean square error of the estimator $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}$ for estimating σ^2?

(A) $\frac{\sigma^4}{n}\left(1-\frac{1}{n}\right)$

(B) $\frac{\sigma^4}{n}\left(1+\frac{1}{n}\right)$

(C) $\frac{(2n+1)\sigma^4}{n^2}$

(D) $\frac{(2n-1)\sigma^4}{n^2}$

(E) $\frac{2\sigma^4(n-1)}{n^2}$

(002) Which one of the following density functions does not belong to the exponential family?

(A) $f(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\theta^3}x^2 e^{-\frac{x^2}{2\theta^2}}, x > 0$

(B) $f(x) = \frac{e^{-\frac{x}{\theta}}}{\theta\left(1+e^{-\frac{x}{\theta}}\right)^2}, -\infty < x < \infty$

(C) $f(x) = \frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}, x \geq 0$

(D) $f(x) = \theta x^{\theta-1}, 0 < x < 1$

(E) $f(x) = \frac{1}{x\sqrt{2\pi\theta}}e^{\frac{-1}{2\theta}(\ln x)^2}, x > 0$

(003) Let $X_1, X_2, \dots, X_n$ be a random sample from Normal distribution with mean $\mu = 0$ and variance σ^2. An estimator $\hat{\sigma}^2 = a\sum X^2$ is unbiased for σ^2 if

(A) $a = \frac{1}{n}$

(B) $a = \frac{1}{n-1}$

(C) $a = n$

(D) $a = n - 1$

(E) $a = 0$

(004) Let $X_1, X_2, \dots, X_n$ be a random sample from the binomial distribution with $E(X) = \theta$, $\mathrm{Var}(X) = \theta(1-\theta)$ and $P(X = x) = \theta^x(1-\theta)^{1-x}$, $x = 0,1$. Then the sample mean

(A) $\bar{X}$ is a sufficient statistic for p.

(B) $\bar{X}$ is an efficient statistic for p.

(C) $\bar{X}$ is a consistent statistic for p.

(D) (A) and (B) are correct but (C) is incorrect.

(E) all (A), (B) and (C) are correct.

(005) If T_1 and T_2 are two most efficient estimators with the same sampling variance ω^2 and the correlation between them is ρ, the variance of $\frac{(T_1+T_2)}{2}$ is equal to

(A) $\frac{(1-\rho)\omega^2}{2}$

(B) $\rho\omega^2$

(C) $\frac{(1+\rho)\omega^2}{2}$

(D) $\frac{(1+\rho)\omega^2}{4}$

(E) $\frac{(1-\rho)\omega^2}{4}$

(006) Let $X_1, X_2, \ldots, X_9$ be a random sample from continuous Uniform distribution with probability density function $f(x) = \frac{1}{3}, \theta - 1 < x < \theta + 2$. What is the bias of sample mean $\bar{X}$?

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{9}{12}$

(D) 1

(E) $\frac{2}{5}$

(007) Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Which one of the following is true for the sum of squares of deviations taken from sample mean?

(A) $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 + n(\bar{X} - \mu)^2$

(B) $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 + 2n(\bar{X} - \mu)^2$

(C) $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - 2n(\bar{X} - \mu)^2$

(D) $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2$

(E) $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i + \mu)^2 + n(\bar{X} - \mu)^2$

(008) The estimator $\hat{\theta} = \sum X^4$ is sufficient for the unknown parameter θ if the probability density function from which the random sample $X_1, X_2, \dots, X_n$ belongs is given as:

(A) $\theta 4^{\theta} x^{\theta-1} e^{-(4x)^{\theta}}, x \geq 0$

(B) $\frac{4^{\theta}}{\Gamma(\theta)} x^{\theta-1} e^{-4x}, x \geq 0$

(C) $4\theta^4 x^3 e^{-(\theta x)^4}, x \geq 0$

(D) $\frac{1}{\Gamma(4)\theta^4} x^3 e^{-\frac{x}{\theta}}, x \geq 0$

(E) $4\theta^{-4} x^{-5} e^{-(\theta x)^{-4}}, x \geq 0$

(009) Let $X_1, X_2, \dots, X_n$ be a random sample from Geometric distribution with probability mass function $P(X = x) = \theta(1-\theta)^{x-1}$, $x = 1, 2, 3, \dots$ .... with $E(X) = \frac{1}{\theta}$ and $\text{Var}(X) = \frac{1-\theta}{\theta^2}$. The sample mean $\bar{X}$ is consistent with

(A) θ

(B) $\frac{1}{1-\theta}$

(C) $\frac{1}{\theta^2}$

(D) θ^2

(E) $\frac{1}{\theta}$

(010) Let $X_1, X_2, \dots, X_n$ be a random sample from continuous Uniform distribution with probability density function $f(x) = \frac{1}{\theta}$, $0 < x < \theta$. The method of moments estimator for θ is given as:

(A) $\frac{\sum X}{n}$

(B) $\frac{2\sum X}{n}$

(C) $\frac{3\sum X^2}{n}$

(D) $\sqrt{\frac{3\sum X^2}{n}}$

(E) $\sqrt{\frac{2\sum X^2}{n}}$

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Q2: (6+9 = 15 pts.) For a probability density function (PDF) denoted by $f(x;\theta)$, the total amount of information contained about θ in the PDF is denoted by I_θ and is given as $I_\theta = E\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2$ under a set of regularity conditions.

(a) Prove that $I_\theta = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right]$.

(b) Let $X_1,\ X_2,\ \dots\ ,\ X_n$ be a random sample from $f(x;\theta)$ for $\theta \in \Omega$. Assuming the regularity conditions hold, let $\hat{\theta} = u(X_1, X_2, \dots, X_n)$ be an estimator of θ, then prove that

$$Var(\hat{\theta}) \geq \frac{[K'(\theta)]^2}{-E\left[\frac{\partial^2}{\partial\theta^2}\ln L(x;\theta)\right]}$$

where $L(x;\theta)$ is the likelihood function and $K(\theta) = E(\hat{\theta})$.

Q2 (b) continues…

Q3: (9 pts.) Let $X_1, X_2, \dots, X_9$ be a random sample from continuous Uniform distribution with probability density function $f(x) = \frac{1}{3}$, $\theta - 1 < x < \theta + 2$. Suppose $\hat{\theta}_1 = \bar{X} - \frac{1}{2}$ and $\hat{\theta}_2 = \frac{X_1 - X_2 + 1}{2} - X_9$ are two unbiased estimators of θ. Find the efficiency of $\hat{\theta}_2$ relative to $\hat{\theta}_1$ i.e. $\frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)}$.

Q3 continues…

Q4: (4+6 = 10 pts.) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function $f(x;\theta) = \frac{2x}{\theta^2}, 0 < x \leq \theta$, zero elsewhere.

(a) Find the MLE for median of distribution.

(b) It is to be noted that the MLE derived in part (a) is a biased estimator for median of distribution. Is it possible to make it unbiased? If yes, do so.

Q5: (6 pts.) Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with probability density function $f(x;\theta) = \theta^2 x e^{-\theta x}$, $x > 0$, zero elsewhere. Derive a minimum variance bound (MVB) estimator for $\frac{2}{\theta}$. Also, find its variance using the formula $\frac{K\prime(\theta)}{A}$.

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Good Luck